CLUSTER VALUES OF ANALYTIC FUNCTIONS ON A BANACH SPACE

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Abstract. We investigate uniform algebras of bounded analytic functions on the unit ball of a complex Banach space. We prove several cluster value theorems, relating cluster sets of a function to its range on the fibers of the spectrum of the algebra. These lead to weak versions of the corona theorem for the open unit balls of c_0 and ℓ_2, in which all but one of the functions comprising the corona data extend to be weak-star continuous on the closed unit ball.

1. Introduction

S. Kakutani [Ka] was perhaps the first to investigate systematically the algebra $H^\infty(\mathbb{D})$ of bounded analytic functions on the open unit disk $\mathbb{D}$ in the complex plane from the point of view of Banach algebras. The cluster set $Cl(f, z_0)$ of $f \in H^\infty(\mathbb{D})$ at a boundary point z_0 of $\mathbb{D}$ is the set of accumulation points of values $f(z)$ as $z \in \mathbb{D}$ tends to z_0. In a collaborative work (of Singer, Wermer, Kakutani, Buck, Royden, Gleason, Arens, and Hoffman), I. J. Schark [Sc] proved that $Cl(f, z_0)$ coincides with the range of the Gelfand transform $\widehat{f}$ of f on the fiber of the spectrum of $H^\infty(\mathbb{D})$ over z_0. For expositions of the circle of ideas related to the cluster value theorem, see [Ho, Chapter 10] and [Ga2].

An analogous cluster value theorem holds for $H^\infty(D)$ for an arbitrary planar domain D [Ga3], and it also holds for polydomains [Ga4] and for smooth strictly pseudoconvex domains [McD] in $\mathbb{C}^n$. The spectrum of $H^\infty(B)$, for B the unit ball of a complex Banach space, was first investigated in [ACG], where it is shown that, even over interior points, fibers are usually highly nontrivial. J. Farmer [Fa] studied the boundary behavior of bounded analytic functions at boundary points of the unit ball of a uniformly convex Banach space, showing that if a function f has a limit at a boundary point w, then $\widehat{f}$ is constant on the fiber over w.

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Our goal is to prove several cluster value theorems for algebras of bounded analytic functions on the open unit ball B of a complex Banach space X. The cluster sets we treat are defined using weak topologies (and not the norm topology on B). In Section 2 we define the algebras of interest and we gather some background material, some of which has been around for some time. In Section 3 we obtain a cluster value theorem at $0 \in B$ for the algebra $A_u(B)$ of uniformly continuous analytic functions on B. In Section 4 we treat this algebra in the special case that B is the open unit ball of Hilbert space, and we obtain a cluster value theorem at all points of the closed unit ball $\bar{B}$. In Section 5 we study the algebra $H^\infty(B)$ of all bounded analytic functions on B, where B is the open unit ball of the Banach space c_0 of null sequences, and we obtain a cluster value theorem at all points of the closed unit ball $\bar{B}^{**}$ of the bidual ℓ_∞ of c_0.

For background on Banach spaces, see [Di]. For background on analytic functions on Banach spaces, see [Mu], [Din] or [Ga5]. For background on uniform algebras, see [Ga1].

2. Background and generalities

Let B be the open unit ball of a complex Banach space X, and let $H^\infty(B)$ be the uniform algebra of bounded analytic functions on B. We denote by $\bar{B}^{**}$ the closed unit ball of the bidual X^{**} of X.

The *cluster set* $Cl_B(f,x)$ of $f \in H^\infty(B)$ at $x \in \bar{B}^{**}$ is the set of all limits of values of f along nets in B converging weak-star to x. Thus $Cl_B(f,x)$ is the intersection of the closures of $f(U \cap B)$, where U ranges over any basis for the weak-star neighborhoods of x. Choosing a basis of convex sets, we see that $Cl_B(f,x)$ is an intersection of a decreasing net of compact connected sets. Thus we have the following.

Lemma 2.1. *Let $f \in H^\infty(B)$. Each cluster set $Cl_B(f,x)$, $x \in \bar{B}^{**}$, is a compact connected set. Further, if $x \in B$, then $f(x) \in Cl_B(f,x)$.*

Example. *If X is an infinite-dimensional Hilbert space, there is a two-homogeneous function f, analytic on the open unit ball B_X of X, such that $|f| \leq 1$ and $Cl_B(f,0)$ coincides with the closed unit disk $\overline{\mathbb{D}}$.*

Indeed, let $\{\lambda_n\}$ be any sequence of complex numbers of absolute value at most 1, such that the λ_n's accumulate on the entire closed unit disk as $n \to \infty$. Define

$f(x) = \sum \lambda_n (x_n)^2$, where the x_n's are the coordinates of x with respect to some orthonormal subset $\{e_n\}$ of X. Since the e_n's converge weakly to 0 as $n \to \infty$, and $f(e_n) = \lambda_n$, the cluster set of f at 0 is the closed unit disk.

Let $A(B)$ denote the algebra of uniform limits on B of polynomials in the functions in X^*. Polynomials in functions in X^* extend to be weak-star continuous on the closed unit ball $\bar{B}^{**}$ of the bidual X^{**} of X, as do their uniform limits. We will view $A(B)$ as a uniform algebra of continuous functions on $\bar{B}^{**}$, with the weak-star topology. The functions in $A(B)$ are analytic on B, and $A(B)$ is a closed subalgebra of $H^\infty(B)$.

It is easy to check that each nonzero complex-valued homomorphism of $A(B)$ is the evaluation homomorphism at some point of $\bar{B}^{**}$. In other words, the spectrum $M_{A(B)}$ of $A(B)$ coincides with $\bar{B}^{**}$.

Let H be an algebra of bounded analytic functions on B containing $A(B)$ and closed under the norm of uniform convergence on B. We are interested specifically in two such algebras, the algebra $H^\infty(B)$ of all bounded analytic functions on B, and the algebra $A_u(B)$ of analytic functions on B that are uniformly continuous with respect to the norm.

We denote by M_H the spectrum of H. The Gelfand theory allows us to regard H as a uniform algebra of functions on M_H. We will denote the Gelfand extension of a function $f \in H$ to M_H by $\widehat{f}$, and view B as a subset of M_H.

The inclusion $A(B) \hookrightarrow H$ induces a natural projection π of M_H onto $M_{A(B)} = \bar{B}^{**}$, so that $\pi(\varphi)$ is simply the restriction of φ to $A(B)$. We define the *fiber* of M_H over $x \in \bar{B}^{**}$ to be $M_x = \pi^{-1}(x)$.

A *cluster value theorem* at $x \in \bar{B}^{**}$ is a theorem that asserts that

$$Cl_B(f, x) = \widehat{f}(M_x), \qquad f \in H. \tag{2.1}$$

One inclusion for this identity is trivial.

Lemma 2.2. *If $f \in H$ and $x \in \bar{B}^{**}$, then $Cl_B(f, x) \subseteq \widehat{f}(M_x)$.*

Proof. If $x \in \bar{B}^{**}$ and $\lambda \in Cl_B(f, x)$, there is a net $\{x_\alpha\}$ in B converging weak-star to x such that $f(x_\alpha) \to \lambda$. Passing to a subnet, we can assume that $x_\alpha \to \varphi$ in M_H. Then $\widehat{f}(\varphi) = \lambda$. Since $\widehat{g}(\varphi) = \lim g(x_\alpha) = g(x)$ for all $g \in A(B)$, $\pi(\varphi) = x$. Thus $\varphi \in M_x$. Hence $\lambda \in \widehat{f}(M_x)$. □

We mention in passing that if the cluster value theorem holds at $x \in \bar{B}^{**}$, then the fiber M_x is connected. This follows from the Shilov idempotent theorem [Ga1, p. 88] and the connectedness of cluster sets. (See [Ho, p.188].)

A *corona theorem* is a theorem that asserts that B is dense in M_H. This occurs if and only if whenever $f_1, \ldots, f_n \in H$ satisfy $|f_1| + \cdots + |f_n| \geq \varepsilon > 0$ on B, there exist $g_1, \ldots, g_n \in H$ such that $f_1 g_1 + \cdots + f_n g_n = 1$. If the corona theorem holds, then evidently the cluster value theorem holds at all points $x \in \bar{B}^{**}$. The following lemma shows how a cluster value theorem may be viewed in some sense as a weak corona theorem. This lemma will be used in Sections 4 and 5.

Lemma 2.3. *The cluster value theorem* (2.1) *holds at every $x \in \bar{B}^{**}$ if and only if whenever $f_1, \ldots, f_{n-1} \in A(B)$ and $f_n \in H$ satisfy $|f_1| + \cdots + |f_n| \geq \varepsilon > 0$ on B, there exist $g_1, \ldots, g_n \in H$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.*

Proof. Suppose the cluster value theorem holds. Let the f_j's satisfy the conditions in the lemma. Suppose the $\widehat{f_j}$'s have a common zero on M_H. Since M_H is fibered over $\bar{B}^{**} = M_{A(B)}$, there is some $x \in \bar{B}^{**}$ such that the $\widehat{f_j}$'s have a common zero on M_x. Then $f_1(x) = \cdots = f_{n-1}(x) = 0$, and $0 \in \widehat{f_n}(M_x)$. By the cluster value theorem, 0 is a cluster value of f_n at x, which contradicts $|f_1| + \cdots + |f_n| \geq \varepsilon > 0$ on B. We conclude that the $\widehat{f_j}$'s have no common zeros on M_H. Thus they belong to no common maximal ideal, and we can solve $\sum f_j g_j = 1$.

For the converse, suppose the cluster value theorem fails at $x \in \bar{B}^{**}$. Choose $g \in H$ such that $\widehat{g}$ has a zero on M_x but $0 \notin Cl_B(g, x)$. Then there is a weak-star open set U in X^{**} containing x such $|g| \geq \varepsilon > 0$ on $U \cap B$. Choose n and functions $L_j \in X^*$, $1 \leq j < n$, such that the functions $f_j = L_j - L_j(x)$ satisfy $\sum |f_j| \geq \varepsilon$ on $B \backslash U$. Then with $f_n = g$ we have $|f_1| + \cdots + |f_n| \geq \varepsilon$ on B. However, $\widehat{f_j} = 0$ on M_x for $1 \leq j \leq n-1$, so $\widehat{f_1}, \ldots, \widehat{f_n}$ have a common zero on M_x, and we cannot solve $\sum f_j g_j = 1$. □

Recall that a point $x \in \bar{B}^{**}$ is a *peak point* for $A(B)$ if there is $g \in A(B)$ such that $g(x) = 1$, and $|g(y)| < 1$ for $y \in \bar{B}^{**}$, $y \neq x$. The function g is said to *peak at* x. (See [Ga1].)

Lemma 2.4. *Let $x \in \bar{B}$, and suppose g is a function in $A(B)$ such that $g(x) = 1$, while $|g|$ is bounded by a constant strictly less than 1 on any subset of B at a positive distance from x. Then g peaks at x. Further, if $f \in H$ is such that $f(y) \to \lambda$ whenever $y \in B$ tends to x in norm, then $\widehat{f} = \lambda$ on M_x.*

Proof. Since $|g| \leq 1$ on B, also $|g| \leq 1$ on $\bar{B}^{**}$. Suppose $y \in \bar{B}^{**}$ is such that $|g(y)| = 1$. If $\{y_\alpha\}$ is a net in B converging weak-star to y, then $|g(y_\alpha)| \to 1$. From the hypothesis on g, we conclude that $y_\alpha \to x$ in norm. Consequently $y = x$, and g peaks at x.

Adding a constant to f, if necessary, we may suppose that $f(y)$ tends to 0 as $y \in B$ tends to x in norm. Then $g^n f \to 0$ uniformly on B as $n \to \infty$. Thus $\widehat{g}^n \widehat{f} \to 0$ uniformly on M_H. Since $\widehat{g} = 1$ on M_x, this can occur only when $\widehat{f} = 0$ on M_x. □

Corollary 2.5. *Suppose $x \in \bar{B}$ is a peak point for $A(B)$. If for each $f \in H$, $f(y)$ has a limit whenever $y \in B$ tends to x in norm, then the fiber M_x reduces to one point, $M_x = \{x\}$.*

Proof. Every function in $\widehat{H}$ is constant on M_x. □

3. The algebra $A_u(B)$

Recall that $A_u(B)$ denotes the algebra of bounded analytic functions on B that are uniformly continuous with respect to the norm of X. The functions in $A(B)$ are norm uniformly continuous on the closed unit ball $\bar{B}$ of X. The example of the 2-homogeneous polynomial given in the preceding section shows that functions in $A_u(B)$ are not necessarily weak-star continuous on $\bar{B}$.

An *m-homogeneous polynomial* on X is the restriction to the diagonal of a bounded m-linear functional. A *polynomial* on X is a finite linear combination of m-homogeneous polynomials for $m \geq 0$. Any polynomial on X is uniformly continuous on B, and the algebra $A_u(B)$ coincides with the uniform limits on $\bar{B}$ of the polynomials. (See [Ga4].)

Our goal in this section is to prove the following theorem.

Theorem 3.1. *If X is a Banach space with a shrinking 1-unconditional basis, then the cluster value theorem holds for $A_u(B)$ at $x = 0$,*

$$Cl_B(f, 0) = \widehat{f}(M_0), \qquad f \in A_u(B).$$

We begin with some lemmas on polynomials.

Suppose Y is a subspace of X of codimension 1. Let L be a continuous linear functional whose kernel is Y, and let $e \in X$ satisfy $L(e) = 1$. Then $P(x) = x - L(x)e$ defines a projection P of X onto Y parallel to e.

Lemma 3.2. *If f is a polynomial of degree n on X, then f can be expressed as $f(x) = f(P(x)) + L(x)g(x)$, where g is a polynomial of degree $n-1$ on X.*

Proof. We may assume that f is n-homogeneous. Then f is the restriction to the diagonal of a symmetric n-linear form F on X, that is, $f(x) = F(x, \ldots, x)$. Setting $y = P(x)$ and $t = L(x)$, we have $f(x) = F(y + te, \ldots, y + te) = F(y, \ldots, y) + tnF(y, \ldots, y, e) + t^2[n(n-1)/2]F(y, \ldots, y, e, e) + \cdots + t^n F(e, \ldots, e)$. We define a function g on X by $g(x) = g(y+te) = nF(y, \ldots, y, e)+[n(n-1)/2]F(y, \ldots, y, te, e)+ \cdots + F(te, \ldots, te, e)$. Then $f(x) = f(P(x))+L(x)g(x)$. Since y and t depend linearly and boundedly on x, g is a polynomial on X, and g has degree $n-1$. □

Lemma 3.3. *Let P be a projection onto a closed subspace Y of X of finite codimension. Then any polynomial f of degree n on X can be expressed in the form $f(x) = f(P(x)) + L_1(x)g_1(x) + \cdots + L_m(x)g_m(x)$, where the g_j's are polynomials of degree $n-1$ on X, and the L_j's are continuous linear functionals on X.*

Proof. This follows from repeated application of the preceding lemma. □

Lemma 3.4. *Let P be a norm-one projection of X onto a closed subspace Y of X of finite codimension. If $\varphi \in M_0$, then $\widehat{f}(\varphi) = \widehat{f \circ P}(\varphi)$ for all $f \in A_u(B)$.*

Proof. Since $\varphi \in M_0$, $\widehat{L}(\varphi) = L(0) = 0$ for all $L \in X^*$. In view of the decomposition of the preceding lemma and the multiplicativity and linearity of φ, we then obtain $\widehat{f}(\varphi) = \widehat{f \circ P}(\varphi)$ for all polynomials f. Since P is a norm-one projection, the equality persists for the uniform limits of polynomials on $\bar{B}$, that is, for all functions in $A_u(B)$. □

Lemma 3.5. *Suppose each weak neighborhood of 0 in B contains the unit ball a subspace of finite codimension with a norm-one projection. Then the cluster value theorem holds for $A_u(B)$ at $x = 0$.*

Proof. Suppose that $0 \notin Cl_B(f, 0)$. We must show that $0 \notin \widehat{f}(M_0)$. Since $0 \notin Cl_B(f, 0)$, there is $\delta > 0$ and a weak neighborhood U of 0 in X such that $|f| \geq \delta$ on $U \cap B$. By hypothesis there is a norm-one projection P of X onto a closed subspace Y of X of finite codimension such that $Y \subset U$. Then $|f \circ P| \geq \delta$ on $X \cap B$, and consequently $f \circ P$ is invertible in $A_u(B)$. Hence $\widehat{f \circ P} \neq 0$ on the spectrum of $A_u(B)$. From the preceding lemma, we then obtain, $\widehat{f} \neq 0$ on M_0, that is, $0 \notin \widehat{f}(M_0)$. □

Proof of Theorem 3.1. Let $\{e_n\}$ be a shrinking 1-unconditional basis for X. Then for each n, the operator $P_n : x = \sum a_k e_k \to \sum_{k\geq n} a_k e_k$ is a norm-one projection. The sets $U_{\varepsilon,m} = \{a = \sum a_k e_k \in B : |a_k| < \varepsilon, 1 \leq k \leq m\}$ form a basis of weak neighborhoods of 0 in B, each weak neighborhood of 0 contains the unit ball of $P_n(X)$ for some large n, and the preceding lemma applies. □

Theorem 3.1 applies in particular to Hilbert space. The proof works in somewhat more generality. For instance it applies to spaces with a shrinking 1-unconditional finite dimensional decomposition. For example, a c_0 or ℓ_p-sum $(1 < p < \infty)$ of finite dimensional spaces E_n whose Gordon-Lewis constants (see Chapter 17 of [DJT]) go to ∞ with n has such a decomposition but cannot have an unconditional basis. We do not know whether the theorem holds for all Banach spaces.

4. The cluster value theorem for Hilbert space

In this section, we take X to be a Hilbert space, and H to be the algebra $A_u(B)$. Since X is reflexive, the spectrum of $A(B)$ is the closed unit ball $\bar{B}$ of X, with the weak topology. The spectrum of $A_u(B)$ is fibered over $\bar{B}$. Our goal in this section is to prove the following theorem.

Theorem 4.1. *If X is a Hilbert space, then the cluster value theorem holds for $A_u(B)$ at every $x \in \bar{B}$,*

$$Cl_B(f, x) = \widehat{f}(M_x), \qquad f \in A_u(B_X), x \in \bar{B}.$$

Corollary 4.2. *Let B be the open unit ball of a Hilbert space. If $f_1, \ldots, f_{n-1} \in A(B)$ and $f_n \in A_u(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \varepsilon > 0$ on B, then there exist $g_1, \ldots, g_n \in A_u(B)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.*

The case of finite-dimensional Hilbert space is trivial, since $A(B) = A_u(B)$. We focus on an infinite-dimensional Hilbert space. The unit ball B of X has a transitive group of automorphisms, and we use these to transfer the cluster value theorem at 0 to other points of B.

Lemma 4.3. *An automorphism ϕ of the open unit ball B of Hilbert space X induces an automorphism $f \to f \circ \phi$ of the uniform algebra $A(B)$. Further, ϕ extends to a homeomorphism of the spectrum of $A(B)$, that is, to a homeomorphism of $\bar{B}$ in the weak topology.*

Proof. For fixed $a \in B$, the formula

$$\beta_a(x) = \frac{1}{1+\sqrt{1-||a||^2}} \left(\frac{x-a}{1-(x|a)} |a \right) a + \sqrt{1-||a||^2} \frac{x-a}{1-(x|a)}, \qquad x \in B,$$

defines an automorphism β_a of B mapping $a \to 0$ and $0 \to -a$. Any automorphism of B mapping a to 0 is the composition of β_a and a unitary operator on X. (See [Re, Proposition 1, p.132].)

By expanding $1/[1-(x|a)]$ as a geometric series $\sum (x|a)^n$ and noting that the series converges uniformly on $\bar{B}$, we see that $\beta_a(x) = g(x)a + h(x)x$, where the functions g and h are in $A(B)$. Let $L \in A(B)$ be a linear functional, that is, $L(x) = (x|z)$ for some $z \in X$. Then $(L \circ \beta_a)(x) = g(x)(a|z) + h(x)(x|z)$, so $L \circ \beta_a \in A(B)$. Since such functions L generate $A(B)$, we see that the composition operator $C : f \to f \circ \beta_a$ leaves $A(B)$ invariant. Since the inverse of β_a is β_{-a}, which also leaves $A(B)$ invariant, C is an automorphism of $A(B)$. Similarly, if U is a unitary operator on X, the composition operator $f \to f \circ U$ is an automorphism of $A(B)$, and in fact $L \circ U$ is linear whenever L is linear. We conclude that if ϕ is any automorphism of B, the composition operator $C_\phi : f \to f \circ \phi$ is an automorphism of $A(B)$. The extension of ϕ to $\bar{B}$ is the restriction of the adjoint operator C_ϕ^* to $\bar{B}$, which is continuous with respect to the weak topology. □

Lemma 4.4. *An automorphism ϕ of the open unit ball B of Hilbert space X induces an automorphism $C_\phi : f \to f \circ \phi$ of the uniform algebra $A_u(B)$. Further, ϕ extends to a homeomorphism $\widehat{\phi}$ of the spectrum $M_{A_u(B)}$, which maps the fiber M_x homeomorphically onto the fiber $M_{\phi(x)}$.*

Proof. From the explicit representation of the automorphisms of B, we see that an automorphism ϕ of B extends to be Lipschitz continuous on $\bar{B}$. Hence the composition operator C_ϕ leaves $A_u(B)$ invariant, and in fact C_ϕ is an automorphism of $A_u(B)$. It follows that the restriction $\widehat{\phi}$ of the adjoint operator C_ϕ^* of C_ϕ to the spectrum $M_{A_u(B)}$ of $A_u(B)$ is a homeomorphism. The induced map $\widehat{\phi}$ is given explicitly by

$$\widehat{f}(\widehat{\phi}(\psi)) = \widehat{f \circ \phi}(\psi), \qquad \psi \in M_{A_u(B)}, f \in A_u(B).$$

Suppose $x \in \bar{B}$ and $\psi \in M_x$. If $f \in A(B)$, then $\widehat{f}(\widehat{\phi}(\psi)) = (\widehat{f \circ \phi})(\psi) = (f \circ \phi)(x) = f(\phi(x))$. Hence $\widehat{\phi}(\psi) \in M_{\phi(x)}$. Since $\widehat{\phi}$ maps the fiber M_x into $M_{\phi(x)}$, and $\widehat{\phi}$ is a homeomorphism of $M_{A_u(B)}$, in fact $\widehat{\phi}$ maps M_x homeomorphically onto $M_{\phi(x)}$. □

Proof of Theorem 4.1. Let $x \in \bar{B}$. If $||x|| = 1$, then the function $g(y) = [1 + (y|x)]/2$ peaks at x, and x is a peak point for $A(B)$. By Corollary 2.5, the fiber M_x of the spectrum of $A_u(B)$ over x consists of only one point, and the cluster value theorem holds trivially for $A_u(B)$ at x.

Suppose on the other hand that $x \in B$. Let ϕ be an automorphism of B such that $\phi(0) = x$. If $f \in A_u(B)$, then clearly $Cl_B(f, x) = Cl_B(C_\phi f, 0)$. By Theorem 3.1, this coincides with $\widehat{C_\phi f}(M_0) = \widehat{f}(\widehat{\phi}(M_0))$, which by the preceding lemma is $\widehat{f}(M_x)$. $\square$

5. The algebra $H^\infty(B)$ on the unit ball of c_0

In this section, we suppose X is the Banach space c_0 of null sequences. In this case, $X^{**} = \ell_\infty$, and B^{**} is the infinite unit polydisk. The algebra $A(B)$ is generated by the linear functionals $x \to \sum a_j x_j$, where $a \in \ell_1$.

A theorem of Littlewood-Bogdanowicz-Pelczynski (see Proposition 1.59 of [Din], or Section 3.4 of [Ga5]) asserts that a bounded m-homogeneous function on c_0 can be approximated uniformly on B by m-homogeneous polynomials of finite type. It follows that the algebra $A_u(B)$ coincides with $A(B)$, and the cluster value theorem holds trivially for $A_u(B)$.

Our goal is to prove a cluster theorem for $H^\infty(B)$. The following example shows that cluster sets of functions in this algebra can be quite large.

Example. *There are functions in $H^\infty(B)$ whose cluster set at 0 contains a disk.*

Indeed, take $r_n < 1$ increasing rapidly to 1, and set $f(x) = \prod (r_n - x_n)/(1 - r_n x_n)$, which is a Blaschke-like product. Clearly $||f|| \leq 1$. Fix μ, $|\mu| < 1$, and choose λ_n such that $\mu = (r_n - \lambda_n)/(1 - r_n \lambda_n)$. Then $|\lambda_n| < 1$, $\lambda_n e_n$ converges weakly to 0, and $f(\lambda_n e_n) \to \mu \prod r_n$.

Theorem 5.1. *If X is the Banach space c_0 of null sequences, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in \bar{B}^{**}$,*

$$Cl_B(f, x) = \widehat{f}(M_x), \qquad f \in H^\infty(B), x \in \bar{B}^{**}.$$

Corollary 5.2. *Let B be the open unit ball of the Banach space c_0 of null sequences. If $f_1, \ldots, f_{n-1} \in A(B)$ and $f_n \in H^\infty(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \varepsilon > 0$ on B, then there exist $g_1, \ldots, g_n \in H^\infty(B)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.*

The cluster theorem at points of B can be easily established by following the line of proof of Theorem 4.1. However, this method does not carry over to arbitrary points in $\bar{B}^{**}$. To handle these points, we use a solution to the $\bar{\partial}$-problem in one complex variable, with control of the dependence of the solution upon analytic parameters. The properties of the solution we will use are summarized in the following lemma. (See Sections II.1 and VIII.10 of [Ga1], or [Ga2], for more details.) We use $\Delta(\zeta_0, \delta)$ to denote the open disk $\{|\zeta - \zeta_0| < \delta\}$ in $\mathbb{C}$.

Lemma 5.3. *Let D be a bounded open subset of $\mathbb{C}$, let $\zeta_0 \in \mathbb{C}$, and let $\delta > 0$. Given $f \in H^\infty(D \cap \Delta(\zeta_0, \delta))$, there are $g \in H^\infty(D)$ and $h \in H^\infty(D \cap \Delta(\zeta_0, \delta))$, given by explicit formulas, such that h extends analytically to $\Delta(\zeta_0, \delta/2)$, and*

$$f(\zeta) = g(\zeta) + (\zeta - \zeta_0)h(\zeta), \qquad \zeta \in D \cap \Delta(\zeta_0, \delta).$$

The supremum norms of g on D and of h on $D \cap \Delta(\zeta_0, \delta)$ can be estimated in terms of δ and the supremum norm of f on D. If f depends analytically on other parameters, so do g and h.

Proof. Let u be a smooth function on $\mathbb{C}$ supported on a compact subset of $\Delta(\zeta_0, \delta)$, such that $u = 1$ in a neighborhood of the closure of $\Delta(\zeta_0, \delta/2)$. Set $f = 0$ off D, and let G be the solution of the $\bar{\partial}$-equation $\bar{\partial} G = f \bar{\partial} u$ which vanishes at ∞. The function G is given explicitly by

$$G(\zeta) = f(\zeta)u(\zeta) + \frac{1}{\pi} \iint f(\lambda) \frac{\partial u}{\partial \bar{\lambda}} \frac{1}{\lambda - \zeta} d\xi d\eta,$$

where $\lambda = \xi + i\eta$. Note that $f - G$ is analytic on $\Delta(\zeta_0, \delta/2)$. We define

$$g(\zeta) = G(\zeta) - (f - G)(\zeta_0), \qquad h(\zeta) = \frac{(f - G)(\zeta) - (f - G)(\zeta_0)}{\zeta - \zeta_0}.$$

Then g and h have the desired properties. □

Proof of Theorem 5.1. Fix $f \in H^\infty(B)$ and $w = (w_1, w_2, \ldots) \in \bar{B}^{**}$. Suppose $0 \notin Cl_B(f, w)$. It will suffice to show that $0 \notin \widehat{f}(M_w)$.

Since 0 is not a cluster value of f at w, there are $c > 0$, $\delta > 0$, and $N \geq 1$ such that if $z \in B$ satisfies $|z_j - w_j| < d$ for $1 \leq j \leq N$, then $|f(z)| \geq c$. For $0 \leq k \leq N - 1$, define

$$U_k = \{z \in B : |z_j - w_j| < d, \quad k + 1 \leq j \leq N\},$$

and set $U_N = B$. Note that $1/f$ is bounded and analytic on U_0.

We claim that for each k, $1 \le k \le N$, there are functions g_k and h_{kj}, $1 \le j \le k$, in $H^\infty(U_k)$ that satisfy

$$f(z)g_k(z) = 1 + (z_1 - w_1)h_{k1}(z) + \cdots + (z_k - w_k)h_{kk}(z), \qquad z \in U_k. \tag{5.1}$$

Once this claim is established, the proof is completed easily as follows. The functions g_N and h_{Nj} belong to $H^\infty(B)$ and satisfy

$$fg_N = 1 + \sum_{j=1}^{N}(z_j - w_j)h_{Nj}.$$

Since each $\widehat{z_j} - \widehat{w_j}$ vanishes on M_w, we obtain $\widehat{f}\widehat{g_N} = 1$ on M_w, and consequently $\widehat{f}$ does not vanish on M_w, as required.

The claim is established by induction on k. The first step, the construction of g_1 and h_{11}, is as follows. We regard $1/f(z_1, z_2, \ldots)$ as a bounded analytic function of z_1 for $|z_1| < 1$, $|z_1 - w_1| < \delta$, with $z_2, z_3, \ldots$ as analytic parameters in the range $|z_j| < 1$, $2 \le j < \infty$ and $|z_j - w_j| < \delta$, $2 \le j \le N$. According to the lemma, we can express

$$\frac{1}{f(z)} = g(z) + (z_1 - w_1)h(z), \qquad z \in U_0,$$

where $g \in H^\infty(U_1)$. If we set $g_1 = g$ and

$$h_{11}(z) = [f(z)g(z) - 1]/(z_1 - w_1), \qquad z \in U_1,$$

then (5.1) is valid for $k = 1$. Note that $h_{11} = -hf$ on U_0. Consequently h_{11} is bounded and analytic on U_0. The defining formula then shows that h_{11} is analytic on all of U_1, and since $|z_1 - w_1| \ge \delta$ on $U_1 \backslash U_0$, h_{11} is bounded on U_1.

Now suppose that $2 \le k \le N$, and that there are functions g_{k-1} and $h_{k-1,j}$ ($1 \le j \le k-1$) that satisfy (5.1) and are appropriately analytic. We apply the lemma to these as functions of z_k, with the other variables regarded as analytic parameters, to obtain decompositions

$$g_{k-1}(z) = g_k(z) + (z_k - w_k)G_k(z)$$

and

$$h_{k-1,j}(z) = h_{k,j}(z) + (z_k - w_k)H_{k,j}(z), \qquad 1 \le j \le k-1,$$

where g_k and the h_{kj}'s are in $H^\infty(U_k)$, and G_k and the H_{kj}'s are in $H^\infty(U_{k-1})$. From the identity (5.1), with k replaced by $k-1$, we obtain

$$fg_k = 1 + \sum_{j=1}^{k-1}(z_j - w_j)h_{kj} + (z_k - w_k)[-fG_k + \sum_{j=1}^{k-1}(z_j - w_j)H_{kj}]$$

on U_{k-1}. We define

$$h_{kk} = [fg_k - 1 - \sum_{j=1}^{k-1}(z_j - w_j)h_{kj}]/(z_k - w_k), \qquad z \in U_k.$$

Then (5.1) is valid. On U_{k-1} we have

$$h_{kk} = -fG_k + \sum_{j=1}^{k-1}(z_j - w_j)H_{kj},$$

so that h_{kk} is bounded and analytic on U_{k-1}. Since $|z_k - w_k| \geq \delta$ on $U_k \backslash U_{k-1}$, we see from the defining formula that $h_{kk} \in H^\infty(U_k)$. This establishes the induction step, and the proof is complete. □

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